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Parabolic Limits of Solutions of Weakly Coupled Parabolic Systems

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In [1], Calderón presented a method for proving the following result in a space of arbitrary dimension: If u is harmonic on $\mathbf{R}^n \times]0, \infty[$, and at each point x of a set $E \subseteq \mathbf{R}^n$, u is bounded in some cone with vertex $(x, 0)$, then u has nontangential limits at almost every point of $E \times \{0\}$. In [6], Hattemer used a similar method to prove the corresponding result for solutions of the heat equation, in which cones were replaced by paraboloids and nontangential limits by "parabolic" limits. In his survey article [3], Chabrowski demonstrated that the method can be routinely extended to solutions of weakly coupled parabolic systems, provided that the coefficients are independent of time. The present paper contains a proof of the result for systems whose coefficients depend on all the variables. It is a nontrivial modification of the method used by the above authors.

We consider the weakly coupled parabolic system

$$L^k(u) = \sum_{i,j=1}^{n} a_{ij}^k(x,t)\frac{\partial^2 u^k}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} b_i^k(x,t)\frac{\partial u^k}{\partial x_i} + \sum_{d=1}^{N} c_d^k(x,t)u^d - \frac{\partial u^k}{\partial t} = 0 \qquad (1)$$

$(k = 1,\dots, N)$ in $\mathbf{R}^n \times [0, c]$, where $0 < c < \infty$ and $u = (u^1,\dots, u^N)$. We make the following assumptions about the coefficients of (1):

(i) The coefficients and derivatives $(\partial/\partial x_i)\, a_{ij}^k$, $\partial^2 a_{ij}^k/\partial x_i \partial x_j$, $(\partial/\partial x_i)\, b_i^k$ are bounded and uniformly Hölder continuous on $R^n \times [0, c]$; $a_{ij}^k = a_{ji}^k$ $(i, j = 1,\dots, n$ and $k = 1,\dots, N)$.

(ii) There exists a positive constant λ such that, for every $\xi \in \mathbf{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}^k(x,t)\,\xi_i \xi_j \geqslant \lambda\,\|\xi\|^2 \qquad (k = 1,\dots, N)$$

whenever $(x, t) \in \mathbf{R}^n \times [0, c]$. Here, and subsequently, $\|\xi\|$ denotes the Euclidean norm of ξ.

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(iii) For every $(x, t) \in \mathbf{R}^n \times]0, c[$ and $d \neq k$, we have $c_k^d(x, t) \geqslant 0$ $(d, k = 1, \dots, N)$.

These hypotheses are the same as those in [4], where several of their consequences are listed. Instead of citing all the various papers which give the consequences we require, we shall refer to the list in [4] wherever possible.

By a *parabolic cone with vertex* x_0, we mean a set $S \subseteq \mathbf{R}^n \times]0, c[$ such that the set $S' = \{(x, \sqrt{t}) : (x, t) \in S\}$ is a cone with vertex x_0. Note that the axis of S' need not be orthogonal to $\mathbf{R}^n \times \{0\}$.

Given $x_0 \in \mathbf{R}^n$, $\alpha > 0$ and $a \in]0, c[$, we use $\Pi_\alpha^a(x_0)$ to denote the particular parabolic cone (or paraboloid)

$$\Pi_\alpha^a(x_0) = \{(x, t) : \|x - x_0\| < \alpha \sqrt{t}, 0 < t < a\}.$$

We say that a solution u of (1) has a *parabolic limit* ω at $x_0 \in \mathbf{R}^n$ if there is a vector $\omega = (\omega^1, \dots, \omega^N)$ of real numbers such that

$$\lim u(x, t) = \omega$$

as $(x, t) \to (x_0, 0)$ from inside $\Pi_\alpha^a(x_0)$ for every $\alpha > 0$.

THEOREM. *Let u be a solution of the system* (1) *on* $\mathbf{R}^n \times]0, c[$. *Let E be a measurable subset of* $\mathbf{R}^n$, *and suppose that, for each $x \in E$, there is a parabolic cone with vertex x on which u is bounded. Then u has a parabolic limit at almost every point of E.*

Proof. We begin by applying Calderón's lemma [1] to each of the functions $(x, t) \mapsto u^k(x, t^2)$ for $k = 1, \dots, N$. Thus we see that given $\alpha > 0$ and $\varepsilon > 0$, there exist a constant $a \in]0, c[$ and a compact set $K \subseteq E$ such that each u^k is bounded on the set

$$G = \bigcup_{x \in K} \Pi_\alpha^a(x), \tag{2}$$

and, if m denotes Lebesgue measure in $\mathbf{R}^n$,

$$m(K) > m(E) - \varepsilon.$$

We shall prove that u has parabolic limits a.e. on K, and the result of the theorem will then follow easily.

Let $\alpha > 0$ and $a \in]0, c[$ be fixed, let G be defined by (2), and let χ denote the characteristic function of G. Let M be a bound for each u^k on G. For each integer $p > 1/a$, put

$$G_p = G \cap (\mathbf{R}^n \times \{1/p\}),$$

and define v_p^k on $\mathbf{R}^n \times]1/p, c[$ by

$$v_p^k(x, t) = \int_{\mathbf{R}^n} \sum_{j=1}^{N} \Gamma_{kj}(x, t; y, 1/p)\, u^j(y, 1/p)\, \chi(y, 1/p)\, dy,$$

where $\{\Gamma_{kj}\}$ is the fundamental matrix of (1). Since u is bounded on G, and there exist positive constants κ and γ such that

$$0 \leqslant \Gamma_{kj}(x, t; y, s) \leqslant \kappa(t-s)^{-n/2} \exp\{-\|x-y\|^2/\gamma(t-s)\} \tag{3}$$

whenever $x, y \in \mathbf{R}^n$, $0 \leqslant s < t \leqslant c$, and $k, j = 1,..., N$ (see [4] for references), it follows from [4, Theorem 1] that v_p is a solution of system (1) on $\mathbf{R}^n \times]1/p, c[$.

Let r be chosen such that $G \subseteq \{(x, t): \|x\| < r,\ 0 < t < a\}$. Then, since $|u^k| \leqslant M$ on G for $k = 1,..., N$, we have

$$\|u^k(\cdot, 1/p)\, \chi(\cdot, 1/p)\|_2 \leqslant \left(\int_{G_p} M^2\, dy\right)^{1/2} \leqslant \left(\int_{\|y\|<r} M^2\, dy\right)^{1/2},$$

so that the sequence $\{u^k(\cdot, 1/p)\, \chi(\cdot, 1/p)\}$ is uniformly bounded in $L^2(\mathbf{R}^n)$. It therefore has a subsequence $\{f_q^k\} = \{u^k(\cdot, 1/p(q))\, \chi(\cdot, 1/p(q))\}$ which converges weakly to some function $f^k \in L^2$. Put

$$v^k(x, t) = \int_{\mathbf{R}^n} \sum_{j=1}^{N} \Gamma_{kj}(x, t; y, 0) f^j(y)\, dy \tag{4}$$

for $(x, t) \in \mathbf{R}^n \times]0, c[$ and $k = 1,..., N$. As with v_p, v is a solution of (1).

We now show that $\{v_{p(q)}^k\}$ converges pointwise to v^k as $q \to \infty$. It follows from (3) that, for each fixed $(x, t) \in \mathbf{R}^n \times]0, c[$ and $k, j = 1,..., N$, the function $y \mapsto \Gamma_{kj}(x, t; y, 0)$ belongs to $L^2(\mathbf{R}^n)$. Therefore, as $q \to \infty$, the weak convergence established above implies that

$$\int_{\mathbf{R}^n} \sum_{j=1}^{N} \Gamma_{kj}(x, t; y, 0)\, f_q^j(y)\, dy \to v^k(x, t) \tag{5}$$

for every $(x, t) \in \mathbf{R}^n \times]0, c[$ and $k = 1,..., N$. Similarly, for each $(x, t) \in \mathbf{R}^n \times]0, c[$, the sequences

$$\left\{\int_{\mathbf{R}^n} \exp(-\|x-y\|^2/\gamma t)\, |f_q^j(y)|\, dy\right\} \tag{6}$$

converge as $q \to \infty$. By the mean value theorem, and the estimates

$$\left| \frac{\partial}{\partial s} \Gamma_{kj}(x, t; y, s) \right| \leqslant \kappa(t-s)^{-(n+2)/2} \exp\{-\|x-y\|^2/\gamma(t-s)\} \tag{7}$$

($x, y \in \mathbf{R}^n$, $0 \leqslant s < t \leqslant c$; see [5, Chap. 9]), for each $(x, t) \in \mathbf{R}^n \times]1/p(q-1), c[$ we have, for some $s \in]0, 1/p(q)[$,

$$\begin{aligned}
&\left| \int_{\mathbf{R}^n} \sum_{j=1}^{N} \{\Gamma_{kj}(x, t; y, 1/p(q)) - \Gamma_{kj}(x, t; y, 0)\} f_q^j(y)\, dy \right| \\
&\quad = \left| \int_{\mathbf{R}^n} \frac{1}{p(q)} \sum_{j=1}^{N} \frac{\partial}{\partial s} \Gamma_{kj}(x, t; y, s) f_q^j(y)\, dy \right| \\
&\quad \leqslant \kappa(t-s)^{-(n+2)/2} p(q)^{-1} \\
&\qquad \times \sum_{j=1}^{N} \int_{\mathbf{R}^n} \exp\{-\|x-y\|^2/\gamma(t-s)\}\, |f_q^j(y)|\, dy \\
&\quad \leqslant \kappa[t-(1/p(q))]^{-(n+2)/2} p(q)^{-1} \\
&\qquad \times \sum_{j=1}^{N} \int_{\mathbf{R}^n} \exp\{-\|x-y\|^2/\gamma t\}\, |f_q^j(y)|\, dy.
\end{aligned}$$

In view of the convergence of the sequences in (6), this last expression tends to zero as $q \to \infty$. This, together with (5), proves that $v_{p(q)}^k \to v^k$ pointwise as $q \to \infty$.

For each integer $p > 1/a$, put

$$w_p = u - v_p$$

on $R^n \times]1/p, c[$. Then $\{w_{p(q)}\}$ converges pointwise to $w = u - v$ as $q \to \infty$. Since each v^k is given by (4), it follows from [2, Theorem 2] that v has parabolic limits a.e. on $\mathbf{R}^n$. We shall prove that w has parabolic limits zero a.e. on K, and for this we require the following auxiliary solution h of the system (1). For each $(x, t) \in \mathbf{R}^n \times]0, c[$ and $k = 1, \ldots, N$, put

$$h^k(x, t) = \delta \int_{\mathbf{R}^n \setminus K} \sum_{j=1}^{N} \Gamma_{kj}(x, t; y, 0)\, dy,$$

where δ is a positive constant to be chosen below. In view of (3), h is a nonnegative solution of (1) and, by [2, Theorem 2], h has parabolic limits zero a.e. on K. We first show that, if δ is sufficiently large, then $h \geqslant M[1 + \kappa N(\pi\gamma)^{n/2}]$ on $\partial G \cap (\mathbf{R}^n \times]0, a[)$, where M is the bound for the u^k on G, and κ, γ are the same as in (3) and (7). We use the estimate

$$\Gamma_{kk}(x, t; y, 0) \geqslant \kappa^{-1} t^{-n/2} \exp\{-\|x-y\|^2/\beta t\} \tag{8}$$

given in [4, (8)]. If $(x, t) \in \partial G \cap (\mathbf{R}^n \times]0, a[)$, then the geometry of G shows that $\{y: \|y - x\| < \alpha\sqrt{t}\} \subseteq \mathbf{R}^n \setminus K$. Therefore (3) and (8) imply that

$$\begin{aligned}
\int_{\mathbf{R}^n\setminus K} \sum_{j=1}^{N} \Gamma_{kj}(x, t; y, 0)\, dy
&\geqslant \int_{\mathbf{R}^n\setminus K} \Gamma_{kk}(x, t; y, 0)\, dy \\
&\geqslant \kappa^{-1} t^{-n/2} \int_{\|y-x\|<\alpha\sqrt{t}} \exp\{-\|x-y\|^2/\beta t\}\, dy \\
&\geqslant \kappa^{-1} t^{-n/2} \exp(-\alpha^2/\beta)\, m(\{y : \|y-x\| < \alpha\sqrt{t}\}) \\
&= \mu,
\end{aligned}$$

say, where μ is a positive constant. Hence $h^k \geqslant M[1 + \kappa N(\pi\gamma)^{n/2}]$ on $\partial G \cap (\mathbf{R}^n \times]0, a[)$ if we choose $\delta = M[1 + \kappa N(\pi\gamma)^{n/2}]/\mu$, which we now do.

The next step is to use this lower bound for the h^k, together with the maximum principle, to show that $|w^k| \leqslant h^k$ on G for $k = 1,..., N$. For each p, the set G_p is open in $\mathbf{R}^n \times \{1/p\}$, so that $u^k\chi$ is continuous on G_p. Therefore, as $(x, t) \to (x_0, 1/p)$, we have $v_p^k(x, t) \to u^k(x_0, 1/p)$ (see [5, Chap. 9]), and hence $w_p^k(x, t) \to 0$, for each $(x_0, 1/p) \in G_p$ and $k = 1,..., N$. Thus

$$\lim\{h^k(x, t) - |w_p^k(x, t)|\} = h(x_0, 1/p) \geqslant 0 \tag{9}$$

as $(x, t) \to (x_0, 1/p)$ for each point $(x_0, 1/p) \in G_p$ and $k = 1,..., N$. Since $|u^k| \leqslant M$ on $\bar{G} \cap (\mathbf{R}^n \times]0, a[)$ for $k = 1,..., N$, we have, by (3),

$$\begin{aligned}
|w_p^k(x, t)| &\leqslant |u^k(x, t)| \\
&\quad + \int_{\mathbf{R}^n} \sum_{j=1}^{N} \Gamma_{kj}(x, t; y, 1/p)\, |u^j(y, 1/p)\, \chi(y, 1/p)|\, dy \\
&\leqslant M + \int_{\mathbf{R}^n} N\kappa[t - (1/p)]^{-n/2} \\
&\quad \times \exp\{-\|x-y\|^2/\gamma[t - (1/p)]\} M\, dy \\
&\leqslant M[1 + N\kappa(\pi\gamma)^{n/2}]
\end{aligned}$$

whenever $(x, t) \in \bar{G} \cap (\mathbf{R}^n \times]1/p, a[)$. It therefore follows from the lower bound for h^k on $\partial G \cap (\mathbf{R}^n \times]0, a[)$ that, as $(x, t) \to (y, s)$ with $(x, t) \in G \cap (\mathbf{R}^n \times]1/p, a[)$

$$\liminf\{h^k(x, t) - |w_p^k(x, t)|\} \geqslant 0 \tag{10}$$

for all $(y, s) \in \partial G \cap (\mathbf{R}^n \times]1/p, a[)$. It follows from (9), (10), and the maximum principle [7, Theorem 2], that

$$|w_p^k(x, t)| \leqslant h^k(x, t)$$

for all $(x,t) \in G \cap (\mathbf{R}^n \times]1/p, a[)$, all p, and $k = 1,..., N$. Restricting our attention to $\{p(q)\}$ and making $q \to \infty$, we deduce that $|w^k| \leqslant h^k$ on G for $k = 1,..., N$.

Since h has parabolic limits zero a.e. on K, and each h^k is bounded away from zero on $\partial G \cap (\mathbf{R}^n \times]0, a[)$, it follows that, for almost every point x_0 in K, any path ending at $(x_0, 0)$ and contained in some paraboloid is eventually contained in G. Therefore, since $|w^k| \leqslant h^k$ on G for $k = 1,..., N$, and h has parabolic limits zero a.e. on K, we deduce that w has similar limits a.e. on K. Since $u = v + w$, and v has parabolic limits a.e. on $\mathbf{R}^n$, it follows that u has parabolic limits a.e. on K. The result now follows easily.

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